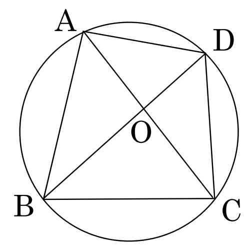
トレミーの定理のベクトルによる証明

1 はじめに

トレミーの定理とは、円に内接する四角形 ABCD が与えられたとき、次の関係式が成り立つことである。
$AD \cdot BC + AB \cdot CD = AC \cdot BD$

2 証明



円に内接する四角形 ABCD を与え、対角線の交点を O とする。
円周角の相等より、$\angle CAD = \angle DBC, \angle ADB = \angle BCA$
これより、

$$\cos\angle CAD = \frac{\overrightarrow{AC}\cdot\overrightarrow{AD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{AD}|} = \frac{\overrightarrow{BD}\cdot\overrightarrow{BC}}{|\overrightarrow{BD}|\cdot|\overrightarrow{BC}|}(=\cos\angle DBC)$$

$$\cos\angle ADB = \frac{\overrightarrow{AD}\cdot\overrightarrow{BD}}{|\overrightarrow{AD}|\cdot|\overrightarrow{BD}|} = \frac{\overrightarrow{AC}\cdot\overrightarrow{BC}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BC}|}(=\cos\angle BCA)$$

また、$\angle AOD = 180^\circ - (\angle CAD + \angle ADB)$ だから、加法定理により、
$\cos\angle AOD = \{180^\circ - (\angle CAD + \angle ADB)\} = -\cos(\angle CAD + \angle ADB)$
$= -\cos\angle CAD \cos ADB + \sin\angle CAD \sin ADB$

よって、$-\cos(\angle CAD + \angle ADB) = -\frac{\overrightarrow{AC}\cdot|\overrightarrow{AD}|^2\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{AD}|^2\cdot|\overrightarrow{BD}|} = +\sin\angle CAD + \sin\angle ADB$

$= -\frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = +\sin\angle CAD + \sin\angle ADB$

$\angle CAD$ を $\angle DBC$ に置き換えると、

$$-\cos(\angle DBC + \angle ADB) = -\frac{\overrightarrow{AD}\cdot|\overrightarrow{BD}|^2\cdot\overrightarrow{BC}}{|\overrightarrow{AD}|\cdot|\overrightarrow{BD}|^2\cdot|\overrightarrow{BC}|} = +\sin\angle CAD + \sin\angle ADB$$

$$= -\frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{|\overrightarrow{AD}|\cdot|\overrightarrow{BC}|} = +\sin\angle CAD + \sin\angle ADB$$

これらより、$\frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = \frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{|\overrightarrow{AD}|\cdot|\overrightarrow{BC}|}$

また、円周角の相等から、$\angle BAC = \angle CDB, \angle ABD = \angle DCA, \angle BOC = 180^\circ - (\angle ABD + \angle DCA)$
よって、同様にして、

$$\frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|}$$

以上から、

$$\frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|} = \frac{\overrightarrow{AD}\cdot\overrightarrow{BC}}{|\overrightarrow{AD}|\cdot|\overrightarrow{BC}|} = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|} = k$$

とすると、

$|\overrightarrow{AD}|\cdot|\overrightarrow{BC}|+|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|=\dfrac{1}{k}(\overrightarrow{AD}\cdot\overrightarrow{BC}+\overrightarrow{AB}\cdot\overrightarrow{CD})$

$=\dfrac{1}{k}\{(\overrightarrow{OA}-\overrightarrow{OD})(\overrightarrow{OC}-\overrightarrow{OB})+(\overrightarrow{OB}-\overrightarrow{OA})(\overrightarrow{OD}-\overrightarrow{OC})\}=\dfrac{1}{k}\{(\overrightarrow{OC}-\overrightarrow{OA})(\overrightarrow{OD}-\overrightarrow{OB})\}$

$=\dfrac{1}{k}\overrightarrow{AC}\cdot\overrightarrow{BD}$

また、$|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|=\dfrac{1}{k}\overrightarrow{AC}\cdot\overrightarrow{BD}$ より、
$|\overrightarrow{AD}|\cdot|\overrightarrow{BC}|+|\overrightarrow{AB}|\cdot|\overrightarrow{CD}|=|\overrightarrow{AC}|\cdot|\overrightarrow{BD}|$　（証明おわり）